MODULI SPACES OF HIGHER SPIN KLEIN SURFACES

SERGEY NATANZON AND ANNA PRATOUSSEVITCH

Abstract. We study connected components of the space of higher spin bundles on hyperbolic Klein surfaces. A Klein surface is a generalisation of a Riemann surface to the case of non-orientable surfaces or surfaces with boundary. The category of Klein surfaces is isomorphic to the category of real algebraic curves. An m-spin bundle on a Klein surface is a complex line bundle whose m-th tensor power is the cotangent bundle. Spaces of higher spin bundles on Klein surfaces are important because of their applications in singularity theory and real algebraic geometry, in particular for the study of real forms of Gorenstein quasi-homogeneous surface singularities. In this paper we describe all connected components of the space of higher spin bundles on hyperbolic Klein surfaces in terms of their topological invariants and prove that any connected component is homeomorphic to the quotient of $\mathbb{R}^d$ by a discrete group. We also discuss applications to real forms of Brieskorn-Pham singularities.

1. Introduction

A complex line bundle $e : L \to P$ on a Riemann surface P, denoted (e, P), is an *m-spin bundle* for an integer $m > 1$ if its m-th tensor power $e^{\otimes m} : L^{\otimes m} \to P$ is isomorphic to the cotangent bundle of P. The classical 2-spin structures on compact Riemann surfaces were introduced by Riemann as theta characteristics and play an important role in mathematics. Their modern interpretation as complex line bundles and classification were given by Atiyah [Ati] and Mumford [Mum], who showed that 2-spin bundles have a topological invariant $\delta = \delta(e, P) \in \{0, 1\}$, the *Arf invariant*, which is determined by the parity of the dimension of the space of sections of the bundle. Moreover, the space $S^2_{g,\delta}$ of 2-spin bundles on Riemann surfaces of genus g with Arf invariant δ, i.e. the space of such pairs (e, P), is homeomorphic to the quotient of $\mathbb{R}^{6g-6}$ by a discrete group of autohomeomorphisms, see [Nat89a, Nat04].

The study of spaces of m-spin bundles for arbitrary m started more recently because of the remarkable connections between the compactified moduli space of m-spin bundles and the theory of integrable systems [Wit], and because of their applications in singularity theory [Dol83, NP11, NP13]. It was shown that for odd m the space of m-spin bundles is connected, while for even m (and $g > 1$) there are two connected components, distinguished by an invariant which generalises the Arf

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invariant [Jar00]. In all cases each connected component of the space of m-spin bundles on Riemann surfaces of genus g is homeomorphic to the quotient of $\mathbb{R}^{6g-6}$ by a discrete group of autohomeomorphisms, see [NP05, NP09]. The homology of these moduli spaces was studied further in [Jar01, JKV, ChZ, FSZ, RW1, RW2, PPZ, SSZ].

The aim of this paper is to determine the topological structure of the space of m-spin bundles on hyperbolic Klein surfaces. A *Klein surface* is a non-orientable topological surface with a maximal atlas whose transition maps are *dianalytic*, i.e. either holomorphic or anti-holomorphic, see [AG]. Klein surfaces can be described as quotients $P/\langle\tau\rangle$, where P is a compact Riemann surface and $\tau : P \to P$ is an anti-holomorphic involution on P. The category of such pairs is isomorphic to the category of Klein surfaces via the relation $(P,\tau) \mapsto P/\langle\tau\rangle$. Under this correspondence the fixed points of τ correspond to the boundary points of the Klein surface. In this paper a Klein surface will be understood as an isomorphism class of such pairs (P,τ). We will only consider connected compact Klein surfaces. The category of connected compact Klein surfaces is isomorphic to the category of irreducible real algebraic curves (see [AG]).

The boundary of a surface $P/\langle\tau\rangle$, if not empty, decomposes into k pairwise disjoint simple closed smooth curves. These closed curves are called *ovals* and correspond to connected components of the set of fixed points P^τ of the involution $\tau : P \to P$. On the real algebraic curve they correspond to connected components of the set of real points.

The *topological type* of the surface $P/\langle\tau\rangle$ is determined by the triple (g,k,ε), where g is the genus of P, k is the number of connected components of the boundary of $P/\langle\tau\rangle$ and $\varepsilon \in \{0,1\}$ with $\varepsilon = 1$ if the surface is orientable and $\varepsilon = 0$ otherwise. The following conditions are satisfied: $1 \leqslant k \leqslant g+1$ and $k \equiv g+1 \pmod 2$ in the case $\varepsilon = 1$ and $0 \leqslant k \leqslant g$ in the case $\varepsilon = 0$. These classification results were obtained by Weichold [Wei]. It is known that the topological type completely determines the connected component of the space of Klein surfaces. Moreover, the space $M_{g,k,\varepsilon}$ of Klein surfaces of the topological type (g,k,ε) is homeomorphic to the quotient of $\mathbb{R}^{3g-3}$ by a discrete subgroup of automorphisms. In addition to the invariants (g,k,ε), it is useful to consider an invariant that we will call the geometric genus of (P,τ). In the case $\varepsilon = 1$ the geometric genus $(g+1-k)/2$ is the number of handles that need to be attached to a sphere with holes to obtain a surface homeomorphic to $P/\langle\tau\rangle$. In the case $\varepsilon = 0$ the geometric genus $[(g-k)/2]$ is half the number of Möbius bands that need to be attached to a sphere with holes to obtain a surface homeomorphic to $P/\langle\tau\rangle$.

An *m-spin bundle on a Klein surface* (P,τ) is a pair (e,β), where $e : L \to P$ is an m-spin bundle on P and $\beta : L \to L$ is an anti-holomorphic involution on L such that $e \circ \beta = \tau \circ e$, i.e. the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "e"] \arrow[d, "\beta"'] & P \arrow[d, "\tau"] \\ L \arrow[r, "e"] & P \end{tikzcd}$$

Spaces of higher spin bundles on Klein surfaces are important because of their applications in singularity theory and real algebraic geometry. We are particularly interested in applications to the classification of real forms of complex singularities. Any Brieskorn-Pham singularity, i.e. singularity of the form $x^a + y^b + z^c = 0$, can be constructed from an m-spin bundle on a Riemann surface P (roughly speaking by contracting the zero section of the bundle) [Mil75, Neu77] and real forms of the singularity correspond to m-spin bundles on Klein surfaces (P, τ). More generally any hyperbolic Gorenstein quasi-homogeneous surface singularity can be constructed from an m-spin bundle on a quotient of the form $\mathbb{H}/\Gamma$, where Γ is a Fuchsian group, possibly with torsion, see [Dol75, Dol77, Dol83]. An extension of the results of our paper to such m-spin bundles will lead to a classification of real forms of hyperbolic Gorenstein quasi-homogeneous surface singularities. The first results in this direction were obtained by H. Riley in her Ph.D. thesis [Ril]. Other classes of complex singularities for which real forms have been studied are simple singularities and cusp singularities, see the survey [W4] as well as [W1, W2, GZ]. See section 5 for more details of applications to singularity theory.

Another important connection is between 2-spin bundles on Klein surfaces and abelian Yang-Mills theory on real tori [OT] and possible generalisations to m-spin bundles.

In this paper we determine the connected components of the space of m-spin bundles on Klein surfaces, i.e. equivalence classes of m-spin bundles on Klein surfaces up to topological equivalence (Definition 3.8). We find the topological invariants that determine such an equivalence class and determine all possible values of these invariants. We also show that every equivalence class is a connected set homeomorphic to the quotient of $\mathbb{R}^n$ by a discrete group, where the dimension n and the group depend on the class. The special case $m = 2$ was studied in [Nat89b, Nat90, Nat99, Nat04].

While 2-spin bundles on a Riemann surface P can be described in terms of quadratic forms on $H_1(P, \mathbb{Z}/2\mathbb{Z})$, for higher spin bundles the situation is more complex. The main innovation of our method is to assign to every m-spin bundle on a Klein surface (P, τ) a function on the set of simple closed curves in P with values in $\mathbb{Z}/m\mathbb{Z}$, called real m-Arf function [NP16]. Thus the problem of topological classification of m-spin bundles on Klein surfaces is reduced to topological classification of real m-Arf functions. We introduce a complete set of topological invariants of real m-Arf functions. We then construct for any real m-Arf function σ a canonical generating set, i.e. a generating set of the fundamental group of P on which σ assumes values determined by the topological invariants.

We will now explain the results in more detail. Let (P, τ) be a Klein surface of type (g, k, ε). In this paper we will consider hyperbolic Klein surfaces (P, τ), i.e. we assume that the underlying Riemann surface P is hyperbolic, $g \geqslant 2$. We will also assume that the geometric genus of (P, τ) is positive, i.e. $k \leqslant g - 2$ if $\varepsilon = 0$ and $k \leqslant g - 1$ if $\varepsilon = 1$.

We show that if m is odd and there exists an m-spin bundle on the Klein surface (P, τ) then $g \equiv 1 \pmod{m}$. Moreover, assuming that m is odd and $g \equiv 1 \pmod{m}$, the space of m-spin bundles on Klein surfaces of type (g, k, ε) is not empty and is connected.

Now let m be even. Consider an m-spin bundle e on the Klein surface (P, τ). A restriction of the bundle e gives a bundle on the ovals. Let K_0 and K_1 be the sets of ovals on which the bundle is trivial and non-trivial respectively. We show that $|K_1|\cdot m/2\equiv 1-g \pmod{m}$.

If m is even and $\varepsilon=0$, the Arf invariant δ of the bundle e and the cardinalities $k_i=|K_i|$ for $i=0,1$ determine a (non-empty) connected component of the space of m-spin bundles on Klein surfaces of type $(g,k_0+k_1,0)$ if and only if

$$k_1\cdot\frac{m}{2}\equiv 1-g \pmod{m}.$$

If m is even and $\varepsilon=1$, the bundle e determines a decomposition of the set of ovals into two disjoint sets, K^0 and K^1, of *similar* ovals (for details see section 3.1). The bundle e induces m-spin bundles on connected components of $P\backslash P^\tau$. The Arf invariant $\tilde{\delta}$ of these induced bundles does not depend on the choice of the connected component of $P\backslash P^\tau$. We also consider the cardinalities $k_i^j=|K_i\cap K^j|$. The invariants $\tilde{\delta}$ and k_i^j for $i,j\in\{0,1\}$ determine a connected component of the space of m-spin bundles on Klein surfaces of type $(g,k_0^0+k_0^1+k_1^0+k_1^1,1)$ if and only if the following conditions are satisfied:

- If $g>k+1$ and $k_0^0+k_0^1\neq 0$ then $\tilde{\delta}=0$.
- If $g>k+1$ and $m\equiv 0 \pmod{4}$ then $\tilde{\delta}=0$.
- If $g=k+1$ and $k_0^0+k_0^1\neq 0$ then $\tilde{\delta}=1$.
- If $g=k+1$ and $m\equiv 0 \pmod{4}$ then $\tilde{\delta}=1$.
- If $g=k+1$ and $k_0^0+k_0^1=0$ and $m\equiv 2 \pmod{4}$ then $\tilde{\delta}\in\{1,2\}$.
- $(k_1^0+k_1^1)\cdot m/2\equiv 1-g \pmod{m}$.

We also show that every connected component of the space of m-spin bundles on Klein surfaces of genus g is homeomorphic to the quotient of $\mathbb{R}^{3g-3}$ by a discrete subgroup of automorphisms which depends on the component (see Theorem 4.3).

The paper is organised as follows:

In section 2 we recall the classification of real m-Arf functions from [NP16]. We determine the topological invariants of real m-Arf functions in section 3. In section 4 we use these topological invariants to describe connected components of the space of m-spin bundles on Klein surfaces. In section 5 we explain the connection between m-spin bundles on Klein surfaces and real forms of complex singularities.

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2. Higher Spin Structures on Klein Surfaces

A *Klein surface* is a topological surface with a maximal atlas whose transition maps are either holomorphic or anti-holomorphic. A *homomorphism* between Klein surfaces is a continuous mapping which is either holomorphic or anti-holomorphic in local charts.

Let us consider pairs (P,τ), where P is a compact Riemann surface and $\tau: P\to P$ is an anti-holomorphic involution on P. For each such pair (P,τ) the quotient $P/\langle\tau\rangle$ is a Klein surface and each isomorphism class of Klein surfaces contains a surface of the form $P/\langle\tau\rangle$. Moreover, two such quotients $P_1/\langle\tau_1\rangle$ and $P_2/\langle\tau_2\rangle$ are isomorphic as Klein surfaces if and only if there exists a biholomorphic map $\psi : P_1 \to P_2$ such that $\psi \circ \tau_1 = \tau_2 \circ \psi$, in which case we say that the pairs (P_1, τ_1) and (P_2, τ_2) are *isomorphic*. Hence from now on instead of Klein surfaces we will consider isomorphism classes of pairs (P, τ). The category of such pairs (P, τ) is isomorphic to the category of real algebraic curves, where fixed points of τ (i.e. boundary points of the corresponding Klein surface) correspond to real points of the real algebraic curve. For example a non-singular plane real algebraic curve given by an equation $F(x, y) = 0$ is the set of real points of such a pair (P, τ), where P is the normalisation and compactification of the surface $\{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0\}$ and τ is given by the complex conjugation, $\tau(x, y) = (\bar{x}, \bar{y})$.

Given two Klein surfaces (P_1, τ_1) and (P_2, τ_2), we say that they are *topologically equivalent* if there exists a homeomorhism $\phi : P_1 \to P_2$ such that $\phi \circ \tau_1 = \tau_2 \circ \phi$.

Let (P, τ) be a Klein surface. The set of fixed points of the involution τ is called the *set of real points* of (P, τ) and denoted by P^τ. We say that (P, τ) is *separating* if the set $P \backslash P^\tau$ is not connected, otherwise we say that it is *non-separating*. The set P^τ decomposes into pairwise disjoint simple closed smooth curves, called *ovals*. Simple closed curves on P which are invariant under the involution τ but do not contain any fixed points of τ are called *twists*. The *topological type* of (P, τ) is the triple (g, k, ε), where g is the genus of the Riemann surface P, k is the number of connected components of the fixed point set P^τ of τ, $\varepsilon = 0$ if (P, τ) is non-separating and $\varepsilon = 1$ otherwise. In this paper we consider hyperbolic surfaces, hence $g \geqslant 2$. Weichold [Wei] classified Klein surfaces up to topological equivalence: Two Klein surfaces are topologically equivalent if and only if they are of the same topological type. A triple (g, k, ε) is a topological type of some Klein surface if and only if either $\varepsilon = 1$, $1 \leqslant k \leqslant g + 1$, $k \equiv g + 1 \pmod 2$ or $\varepsilon = 0$, $0 \leqslant k \leqslant g$. For more detailed discussion of Klein surfaces see [AG, Nat90].

A line bundle $e : L \to P$ on a Riemann surface P is an m-spin bundle (of rank 1) if the m-fold tensor power $L \otimes \cdots \otimes L \to P$ coincides with the cotangent bundle of P. For $m = 2$ we obtain the classical notion of a spin bundle. In [NP05, NP09] we proved that m-spin bundles on P are in 1-1-correspondence with *m-Arf functions*, certain functions on the space $\pi_1^0(P)$ of homotopy classes of simple closed curves on P with values in $\mathbb{Z}/m\mathbb{Z}$ described by simple geometric properties. We introduced topological invariants of m-Arf functions, in particular the *Arf inariant* δ, and described the conditions for existence of an m-Arf function with prescribed values on a generating set of $\pi_1(P)$.

Let (P, τ) be a Klein surface. A classification of m-spin bundles on P that are invariant under τ was given in [NP16]. Such bundles are characterised by the special properties of the corresponding m-Arf functions, called real m-Arf functions. An m-Arf function σ on P is *real* if $\sigma(\tau c) = -\sigma(c)$ for any c and $\sigma(c) = 0$ for any twist c. The mapping that assigns to an m-spin bundle on P the corresponding m-Arf function establishes a 1-1-correspondence between m-spin bundles invariant under τ and real m-Arf functions on P. In [NP16] we determined the conditions for existence of real m-Arf functions with prescribed values on a *symmetric generating set,* which is a generating set of $\pi_1(P)$ which is particularly well adapted to the action of τ. Furthermore we enumerated such real m-Arf functions. For details see section 4.4 in [NP16], in particular Theorems 4.9 and 4.10.

3. Topological Types of Higher Arf Functions on Klein Surfaces

3.1. **Topological Invariants.**

Definition 3.1. Let (P,τ) be a non-separating Klein surface of type $(g,k,0)$. Let m be even. The *topological type* of a real m-Arf function σ on (P,τ) is a tuple (g,δ,k_0,k_1), where g is the genus of P, δ is the m-Arf invariant of σ and k_j is the number of ovals of (P,τ) with the value of σ equal to $j\cdot m/2$.

Real m-Arf functions with even m on separating Klein surfaces have additional topological invariants:

Definition 3.2. Let (P,τ) be a separating Klein surface of type $(g,k,1)$. Let P_1 and P_2 be the connected components of $P\backslash P^\tau$. Let m be even. Let σ be an m-Arf function on (P,τ). We say that two ovals c_1 and c_2 are *similar* with respect to σ, $c_1\sim c_2$, if $\sigma(\ell\cup(\tau\ell)^{-1})$ is odd, where ℓ is a simple path in P_1 connecting a point on c_1 to a point on c_2.

From the definition of m-Arf functions (see Definition 3.4 in [NP16]) it is clear that if $\sigma:\pi_1^0(P)\to\mathbb{Z}/m\mathbb{Z}$ is a real m-Arf function on (P,τ) and m is even, then $(\sigma \pmod 2):\pi_1^0(P)\to\mathbb{Z}/2\mathbb{Z}$ is a real 2-Arf function on (P,τ). Note that two ovals are similar with respect to the m-Arf function σ if and only if they are similar with respect to the 2-Arf function $(\sigma \pmod 2)$, hence we obtain using [Nat04], Theorem 3.3:

Proposition 3.1. *Similarity of ovals is well-defined. Similarity is an equivalence relation on the set of all ovals with at most two equivalence classes.*

Definition 3.3. Let (P,τ) be a separating Klein surface of type $(g,k,1)$. Let P_1 and P_2 be the connected components of $P\backslash P^\tau$. Let m be even. Let us choose one similarity class of ovals. The *topological type* of a real m-Arf function σ on (P,τ) is a tuple

$$(g,\tilde\delta,\{(k_0^0,k_1^0),(k_0^1,k_1^1)\}),$$

where g is the genus of P, $\tilde\delta$ is the m-Arf invariant of $\sigma|_{P_1}$, k_j^0 is the number of ovals in the chosen similarity class with the value of σ equal to $j\cdot m/2$ and $k_j^1=k_j-k_j^0$ is the number of ovals in the other similarity class with the value of σ equal to $j\cdot m/2$. The invariants k_j^i depend on the choice of a similarity class of ovals, choosing the other similarity class leads to the swap $(k_0^0,k_1^0)\leftrightarrow(k_0^1,k_1^1)$.

Definition 3.4. Let (P,τ) be a Klein surface of type (g,k,ε). Let m be odd. The *topological type* of a real m-Arf function σ on (P,τ) is a pair (g,k), where g is the genus of P and k the number of ovals of (P,τ).

Proposition 3.2. *If there exists a real m-Arf function of the topological type t on a Klein surface of type (g,k,ε), $g\geqslant 2$, then t satisfies the following conditions:*

1) Case $\varepsilon=0$, $m\equiv 0 \pmod 2$, $t=(g,\delta,k_0,k_1)$:
$k_1\cdot m/2\equiv 1-g \pmod m$.

2) Case $\varepsilon=1$, $m\equiv 0 \pmod 2$, $t=(g,\tilde\delta,(\{(k_0^0,k_1^0),(k_0^1,k_1^1)\})$:
Let $k_j=k_j^0+k_j^1$ for $j=0,1$.

- *(a) If $g>k+1$ and $m\equiv 0 \pmod 4$ then $\tilde\delta=0$.*
- *(b) If $g>k+1$ and $k_0\neq 0$ then $\tilde\delta=0$.*
- *(c) If $g=k+1$ and $m\equiv 0 \pmod 4$ then $\tilde\delta=1$.*
- *(d) If $g=k+1$ and $k_0\neq 0$ then $\tilde\delta=1$.*
- *(e) If $g=k+1$, $m\equiv 2 \pmod 4$ and $k_0=0$ then $\tilde\delta\in\{1,2\}$.*

(f) $k_1 \cdot m/2 \equiv 1 - g \pmod{m}$.

3) Case $m \equiv 1 \pmod{2}$*,* $t = (g, k)$*:*

$g \equiv 1 \pmod{m}$.

Proof. Let (P, τ) be a Klein surface of type (g, k, ε), $g \geqslant 2$. Let σ be a real m-Arf function of the topological type t on (P, τ). Let $c_1, \dots, c_k$ be the ovals of (P, τ).

1) Case $\varepsilon = 0$, $m \equiv 0 \pmod{2}$, $t = (g, \delta, k_0, k_1)$: By the definition of the k_j, the tuple $(\sigma(c_1), \dots, \sigma(c_k))$ is a permutation of zero repeated k_0 times and $m/2$ repeated k_1 times, hence $\sum\limits_{i=1}^{k} \sigma(c_i) \equiv k_1 \cdot m/2 \pmod{m}$. On the other hand Theorem 4.9(1) in [NP16] implies $\sum\limits_{i=1}^{k} \sigma(c_i) \equiv 1 - g \pmod{m}$. Hence $k_1 \cdot m/2 \equiv 1 - g \pmod{m}$.
2) Case $\varepsilon = 1$, $m \equiv 0 \pmod{2}$, $t = (g, \tilde{\delta}, \{(k_0^0, k_1^0), (k_0^1, k_1^1)\})$: Let P_1 and P_2 be the connected components of $P \backslash P^\tau$. Each of these components is a surface of genus $\tilde{g} = (g+1-k)/2$ with k holes. If σ is a real m-Arf function of the topological type $(g, \tilde{\delta}, \{(k_0^0, k_1^0), (k_0^1, k_1^1)\})$ on (P, τ), then $\sigma|_{P_1}$ is an m-Arf function on a surface of genus $\tilde{g}$ with k holes with the values on the holes equal to zero repeated k_0 times and $m/2$ repeated k_1 times.
   - Theorem 4.3(b) in [NP16] implies that if $\tilde{g} > 1$ and $\sigma(c_i) \equiv 0 \pmod{2}$ for some i then $\tilde{\delta} = 0$. Note that $\tilde{g} > 1$ if and only if $g > k + 1$. If $m \equiv 0 \pmod{4}$ then all $\sigma(c_i)$ are even since both 0 and $m/2$ are even, therefore $\tilde{\delta} = 0$. If $k_0 \neq 0$ then $\sigma(c_i) = 0$ for some i, hence $\sigma(c_i)$ is even for some i, therefore $\tilde{\delta} = 0$. However, if $m \equiv 2 \pmod{4}$ and $k_0 = 0$ then all $\sigma(c_i) = m/2$ are odd, hence no conclusion can be made about $\tilde{\delta}$. Thus we can rewrite the condition as follows: If $g > k + 1$ and ($m \equiv 0 \pmod{4}$ or $k_0 \neq 0$) then $\tilde{\delta} = 0$.
   - Theorem 4.3(c) in [NP16] implies that in the case $\tilde{g} = 1$ the Arf invariant $\tilde{\delta}$ is a divisor of $\gcd(m, \sigma(c_1)+1, \dots, \sigma(c_k)+1)$. Note that $\tilde{g} = 1$ if and only if $g = k+1$. If $k_0 \neq 0$ then $\sigma(c_i) = 0$ for some i, hence $\tilde{\delta}$ is a divisor of $\gcd(m, 1, \dots)$, therefore $\tilde{\delta} = 1$. If $k_0 = 0$ then $\sigma(c_i) = m/2$ for all i, hence $\tilde{\delta}$ is a divisor of $\gcd\left(m, \frac{m}{2} + 1\right)$. For $m \equiv 0 \pmod{4}$ we have $\gcd\left(m, \frac{m}{2} + 1\right) = 1$, hence $\tilde{\delta} = 1$. For $m \equiv 2 \pmod{4}$ we have $\gcd\left(m, \frac{m}{2} + 1\right) = 2$, hence $\tilde{\delta} \in \{1, 2\}$. Therefore we can rewrite the condition as follows: If $g = k + 1$ and ($m \equiv 0 \pmod{4}$ or $k_0 \neq 0$) then $\tilde{\delta} = 1$. If $g = k + 1$, $m \equiv 2 \pmod{4}$ and $k_0 = 0$ then $\tilde{\delta} \in \{1, 2\}$.
   - Theorem 4.3(d) in [NP16] implies that $\sigma(c_1) + \cdots + \sigma(c_k) \equiv (2 - 2\tilde{g}) - k \pmod{m}$. Note that $\sigma(c_1) + \cdots + \sigma(c_k) = k_1 \cdot m/2$ and $(2 - 2\tilde{g}) - k = 1 - g$. Hence we can rewrite the condition as follows: $k_1 \cdot m/2 \equiv 1 - g \pmod{m}$.
3) Case $m \equiv 1 \pmod{2}$, $t = (g, k)$: Theorem 4.10(1) in [NP16] implies $g \equiv 1 \pmod{m}$.

□

Proposition 3.3. *Let* (P, τ) *be a Klein surface of type* $(g, k, 1)$*,* $g \geqslant 2$*, and let* m *be even. Let* σ *be an* m*-Arf function of type* $(g, \tilde{\delta}, \{(k_0^0, k_1^0), (k_0^1, k_1^1)\})$ *on* (P, τ)*. Then the Arf invariant* $\delta \in \{0, 1\}$ *of* σ *is given by*

$$\delta \equiv k_0^0 \equiv k_0^1 \pmod{2} \quad \textit{if} \quad m \equiv 2 \pmod{4},$$
$$\delta \equiv k_0^0 + k_1^0 \equiv k_0^1 + k_1^1 \pmod{2} \quad \textit{if} \quad m \equiv 0 \pmod{4}.$$

Proof. Consider an m-Arf function σ of type $(g, \tilde{\delta}, \{(k_0^0, k_1^0), (k_0^1, k_1^1)\})$ on (P, τ). Let $c_1, \dots, c_k$ be the ovals of (P, τ). We choose a symmetric generating set

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a_1', b_1', \dots, a_{\tilde{g}}', b_{\tilde{g}}', c_1, \dots, c_{k-1}, d_1, \dots, d_{k-1}).$$

of $\pi_1(P)$. Set $\gamma_i = \sigma(c_i)$ for $i = 1, \dots, k$ and $\delta_i = \sigma(d_i)$ for $i = 1, \dots, k-1$. We can assume without loss of generality that the chosen similarity class contains the oval c_k (see Definition 3.3). Let $\delta_k = 1$. For $\alpha, \beta \in \{0, 1\}$ let A_α^β be the subsets of $\{1, \dots, k\}$ given by

$$A_\alpha^\beta = \{i \mid \gamma_i = \alpha \cdot m/2, \delta_i \equiv 1 - \beta \pmod 2\}.$$

Then $k \in A_0^0 \cup A_1^0$. Note that $|A_\alpha^\beta| = k_\alpha^\beta$. According to Theorem 4.9(4) in [NP16], the Arf invariant δ of σ is given by

$$\delta \equiv \sum_{i=1}^{k-1} (1 - \gamma_i)(1 - \delta_i) \pmod 2.$$

Weichold's classification of Klein surfaces implies $k \equiv g + 1 \pmod 2$. If $m \equiv 2 \pmod 4$, then

$$\sum_{i=1}^{k-1} (1 - \gamma_i)(1 - \delta_i) \equiv |A_0^1 \cap \{1, \dots, k-1\}| \equiv |A_0^1| \equiv k_0^1 \pmod 2.$$

In this case $m/2$ is odd, hence condition $k_1 \cdot m/2 \equiv 1 - g \pmod m$ can be reduced modulo 2 to $k_1 \equiv 1 - g \pmod 2$. Using $k \equiv g + 1 \pmod 2$ we obtain

$$k_0 = k - k_1 \equiv (g + 1) - (1 - g) \equiv 0 \pmod 2,$$

i.e.

$$k_0^1 = k_0 - k_0^0 \equiv k_0^0 \pmod 2.$$

If $m \equiv 0 \pmod 4$, then

$$\sum_{i=1}^{k-1} (1 - \gamma_i)(1 - \delta_i) \equiv |(A_0^1 \cup A_1^1) \cap \{1, \dots, k-1\}| \equiv |A_0^1 \cup A_1^1| \equiv k_0^1 + k_1^1 \pmod 2.$$

In this case $m/2$ is even, hence condition $k_1 \cdot m/2 \equiv 1 - g \pmod m$ can be reduced modulo 2 to $0 \equiv 1 - g (\mathrm{mod}\, 2)$, hence g is odd, so that $k \equiv g + 1 \pmod 2$ is even. Therefore

$$k_0^1 + k_1^1 = k - (k_0^0 + k_1^0) \equiv k_0^0 + k_1^0 \pmod 2.$$

□

3.2. **Canonical Symmetric Generating Sets.** For a Klein surface (P, τ) we introduced in [NP16] symmetric generating sets of $\pi_1(P)$. These generating sets have certain symmetry with respect to the action of τ. In this section we will construct for any real m-Arf function σ a standard generating set of $\pi_1(P)$ on which σ assumes prescribed values determined by the topological invariants of σ. We will call such a generating set canonical for σ. For the convenience of the reader we will first recall the definition of a standard generating set. The following fact is well known, see for example [Nat04, Nat75, Nat78] and [B]:

Proposition 3.4. *Let (P, τ) be a Klein surface of the topological type (g, k, ε). Let $c_1, \dots, c_k$ be the ovals of (P, τ). In the case $\varepsilon = 0$ we can choose for any n with $k + 1 \leqslant n \leqslant g + 1$ and $n \equiv g + 1 \pmod 2$ twists $c_{k+1}, \dots, c_n$ such that the complement of the curves $c_1, \dots, c_n$ in P consists of two components P_1 and P_2.*

In the case $\varepsilon = 1$ we can take $n = k$. Each of the components P_1 and P_2 is a surface of genus $\tilde{g} = (g + 1 - n)/2$ with n holes. We will refer to P_1 and P_2 as a decomposition of (P, τ) in two halves. Note that such a decomposition is unique if (P, τ) is separating, but is not unique if (P, τ) is non-separating since the twists $c_{k+1}, \ldots, c_n$ can be chosen in different ways.

Definition 3.5. Let (P, τ) be a Klein surface and $c_1, \ldots, c_n$ invariant closed curves as in Proposition 3.4 such that the complement of the curves $c_1, \ldots, c_n$ in P consists of two components P_1 and P_2. For two invariant closed curves c_i and c_j, a *bridge* between c_i and c_j is a curve of the form

$$r_i \cup (\tau\ell)^{-1} \cup r_j \cup \ell,$$

where:

- ℓ is a simple path in P_1 starting at a point on c_j and ending at a point on c_i.
- r_i is the path along c_i from the end point of ℓ to the end point of $\tau\ell$. (If c_i is an oval then the path r_i consists of one point.)
- r_j is the path along c_j from the starting point of $\tau\ell$ to the starting point of ℓ. (If c_j is an oval then the path r_j consists of one point.)

Figure 1 shows the shapes of the bridges for different types of invariant curves. The bridges are shown in bold. The bold arrows on the bold lines show the direction of the bridges, while the thinner arrows near the lines show the directions of the paths c_i, c_j, r_i, r_j, ℓ and $\tau\ell$.

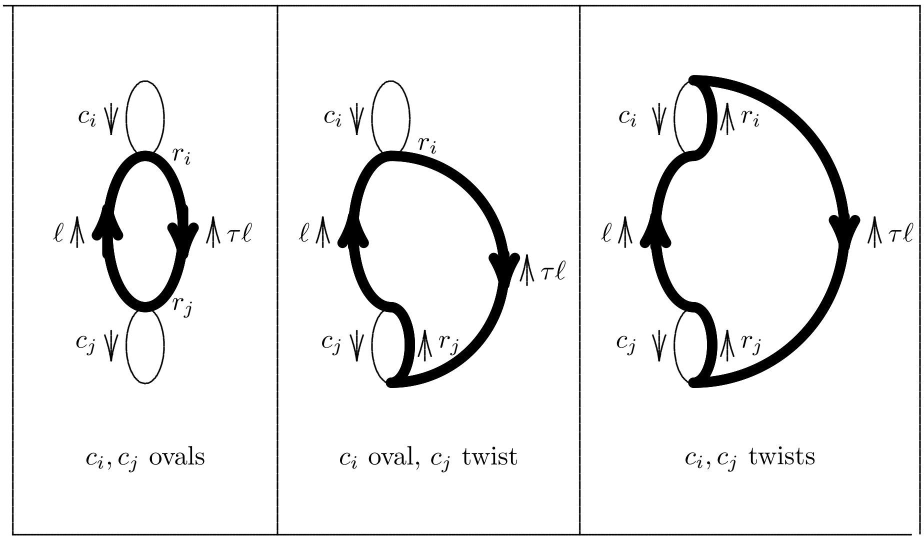

Figure 1: Bridges

Definition 3.6. Let (P, τ) be a Klein surface of the topological type (g, k, ε). A *symmetric generating set* of $\pi_1(P)$ is a generating set of the form

$$(a_1, b_1, \ldots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \ldots, a'_{\tilde{g}}, b'_{\tilde{g}}, c_1, \ldots, c_{n-1}, d_1, \ldots, d_{n-1}),$$

where

- $n = k$ if $\varepsilon = 1$.

- $k+1 \leqslant n \leqslant g+1$ and $n \equiv g+1 \pmod 2$ if $\varepsilon = 0$.
- $c_1, \dots, c_k$ are the ovals of (P, τ).
- $c_{k+1}, \dots, c_{n-1}$ are twists (in the case $\varepsilon = 0$).
- There exists an invariant closed curve c_n such that the complement of the curves $c_1, \dots, c_n$ in P consists of two components P_1 and P_2. The invariant curve c_n is an oval if $\varepsilon = 1$ and a twist if $\varepsilon = 0$.
- $(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n)$ is a generating set of $\pi_1(P_1)$.
- $a'_i = (\tau a_i)^{-1}$ and $b'_i = (\tau b_i)^{-1}$ for $i = 1, \dots, \tilde{g}$.
- $d_1, \dots, d_{n-1}$ are closed curves which only intersect at the base point, such that d_i is homotopic to a bridge between c_i and c_n,

Note that $\tau c_i = c_i$ and $\tau d_i = c_i^{|c_i|} d_i^{-1} c_n^{|c_n|}$, where $|c_j| = 0$ if c_j is an oval and $|c_j| = 1$ if c_j is a twist.

Definition 3.7. Let (P, τ) be a Klein surface of type (g, k, ε), $g \geqslant 2$, and σ a real m-Arf function σ of the topological type t on (P, τ). Let

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, c_1, \dots, c_{n-1}, d_1 \dots, d_{n-1})$$

be a symmetric generating set of $\pi_1(P)$ and

$$\alpha_i = \sigma(a_i),\ \beta_i = \sigma(b_i),\ \alpha'_i = \sigma(a'_i),\ \beta'_i = \sigma(b'_i),\ \gamma_i = \sigma(c_i),\ \delta_i = \sigma(d_i).$$

We say that $\mathcal{B}$ is *canonical* for the m-Arf function σ if

- Case $\varepsilon = 0$, $m \equiv 0 \pmod 2$, $t = (g, \delta, k_0, k_1)$:
$$(\alpha_1, \beta_1, \dots, \alpha_{\tilde{g}}, \beta_{\tilde{g}}) = (\alpha'_1, \beta'_1, \dots, \alpha'_{\tilde{g}}, \beta'_{\tilde{g}}) = (0, 1, 1, \dots, 1) \text{ if } \tilde{g} \geqslant 2,$$
$$(\alpha_1, \beta_1) = (\alpha'_1, \beta'_1) = (1, 0) \text{ if } \tilde{g} = 1,$$
$$\gamma_1 = \dots = \gamma_{k_0} = 0, \quad \gamma_{k_0+1} = \dots = \gamma_k = m/2, \quad \gamma_{k+1} = \dots = \gamma_{n-1} = 0,$$
$$\delta_1 = \dots = \delta_{n-1} = 1 - \delta.$$
- Case $\varepsilon = 1$, $m \equiv 0 \pmod 2$, $t = (g, \tilde{\delta}, \{(k_0^0, k_1^0), (k_0^1, k_1^1)\})$:
$$(\alpha_1, \beta_1, \dots, \alpha_{\tilde{g}}, \beta_{\tilde{g}}) = (\alpha'_1, \beta'_1, \dots, \alpha'_{\tilde{g}}, \beta'_{\tilde{g}}) = (0, 1 - \tilde{\delta}, 1, \dots, 1) \text{ if } \tilde{g} \geqslant 2;$$
$$(\alpha_1, \beta_1) = (\alpha'_1, \beta'_1) = (\tilde{\delta}, 0) \text{ if } \tilde{g} = 1;$$
$$\gamma_1 = \dots = \gamma_{k_0} = 0, \quad \gamma_{k_0+1} = \dots = \gamma_{k-1} = m/2;$$
In the case $k_1 \geqslant 1$: $\delta_1 = \dots = \delta_{k_0^1} = 0, \quad \delta_{k_0^1+1} = \dots = \delta_{k_0} = 1,$
$$\delta_{k_0+1} = \dots = \delta_{k_0+k_1^1} = 0, \quad \delta_{k_0+k_1^1+1} = \dots = \delta_{k-1} = 1;$$
In the case $k_1 = 0$: $\delta_1 = \dots = \delta_{k_0^1} = 0, \quad \delta_{k_0^1+1} = \dots = \delta_{k-1} = 1.$
- Case $m \equiv 1 \pmod 2$, $t = (g, k)$:
$$(\alpha_1, \beta_1, \dots, \alpha_{\tilde{g}}, \beta_{\tilde{g}}) = (\alpha'_1, \beta'_1, \dots, \alpha'_{\tilde{g}}, \beta'_{\tilde{g}}) = (0, 1, 1, \dots, 1) \text{ if } \tilde{g} \geqslant 2,$$
$$(\alpha_1, \beta_1) = (\alpha'_1, \beta'_1) = (1, 0) \text{ if } \tilde{g} = 1,$$
$$\gamma_1 = \dots = \gamma_{n-1} = 0,$$
$$\delta_1 = \dots = \delta_{n-1} = 0.$$

Lemma 3.5. *Let (P, τ) be a Klein surface of type (g, k, ε), $g \geqslant 2$. Let the geometric genus of (P, τ) be positive, i.e. $k \leqslant g - 1$ if $\varepsilon = 1$ and $k \leqslant g - 2$ if $\varepsilon = 0$. In the case $\varepsilon = 1$ let $n = k$. In the case $\varepsilon = 0$ we choose $n \in \{k+1, \dots, g-1\}$ such that $n \equiv g - 1 \pmod 2$. (The assumption that the geometric genus is positive implies $k + 1 \leqslant g - 1$, hence $\{k+1, \dots, g-1\} \neq \varnothing$.) Let $c_1, \dots, c_n$ be invariant closed* *curves as in Proposition 3.4, then bridges $d_1, \dots, d_{n-1}$ as in Definition 3.6 can be chosen in such a way that*

(i) If m is odd, then $\sigma(d_i) = 0$ for $i = 1, \dots, n-1$.
(ii) If m is even and (P, τ) is separating, then $\sigma(d_i) \in \{0, 1\}$ for $i = 1, \dots, n-1$.
(iii) If m is even and (P, τ) is non-separating, then $\sigma(d_1) = \cdots = \sigma(d_{n-1}) \in \{0, 1\}$.

Proof. Let P_1 and P_2 be the connected components of the complement of the closed curves $c_1, \dots, c_n$ in P. Each of these components is a surface of genus $\tilde{g} = (g + 1 - n)/2$ with n holes. The assumption $n \leqslant g - 1$ implies $\tilde{g} \geqslant 1$.

- Consider the real 2-Arf function $(\sigma \pmod 2) : \pi_1^0(P) \to \mathbb{Z}/2\mathbb{Z}$. If m is even and (P, τ) is non-separating, then, according to Lemma 11.2 in [Nat04], we can choose the bridges $d_1, \dots, d_{n-1}$ so that
$$(\sigma \pmod 2)(d_1) = \cdots = (\sigma \pmod 2)(d_{n-1}).$$
This means for the original m-Arf function σ that
$$\sigma(d_1) \equiv \cdots \equiv \sigma(d_{n-1}) \pmod 2.$$
- Let Q_1 be the compact surface of genus $\tilde{g}$ with one hole obtained from P_1 after removing all bridges $d_1, \dots, d_{n-1}$. Let $\tilde{\delta}$ be the Arf invariant of $\sigma|_{Q_1}$. In the case $\tilde{g} \geqslant 2$, Lemma 5.1 in [NP09] implies that we can choose a standard generating set $(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, \tilde{c})$ of $\pi_1(Q_1)$ so that $\sigma(a_1) = 0$. In the case $\tilde{g} = 1$, Lemma 5.2 in [NP09] implies that we can choose a standard generating set $(a_1, b_1, \tilde{c})$ of $\pi_1(Q_1)$ so that $\sigma(b_1) = 0$. Thus for $\tilde{g} \geqslant 1$ there always exists a non-trivial closed curve a in P_1 with $\sigma(a) = 0$, which does not intersect any of the bridges $d_1, \dots, d_{n-1}$. If we replace d_i by $(\tau a)^{-1} d_i a$, then
$$\sigma((\tau a)^{-1} d_i a) = \sigma((\tau a)^{-1}) + \sigma(d_i) + \sigma(a) - 2.$$
Taking into account the fact that $\sigma(a) = 0$ we obtain
$$\sigma((\tau a)^{-1} d_i a) = \sigma(d_i) - 2.$$
Repeating this operation we can obtain $\sigma(d_i) = 0$ for odd m and $\sigma(d_i) \in \{0, 1\}$ for even m.
- Note that the property $\sigma(d_1) \equiv \cdots \equiv \sigma(d_{n-1}) \pmod 2$ (if m is even and (P, τ) is non-separating) is preserved during this process, hence $\sigma(d_1) = \cdots = \sigma(d_{n-1})$ at the end of the process.

□

Proposition 3.6. *Let (P, τ) be a Klein surface of positive geometric genus. For any real m-Arf function on (P, τ) there exists a canonical symmetric generating set of $\pi_1(P)$.*

Proof. Let (g, k, ε) be the topological type of the Klein surface (P, τ). Let σ be a real m-Arf function on (P, τ). Let $c_1, \dots, c_n$ be invariant closed curves as in Proposition 3.4.

- If $m \equiv 0 \pmod 2$ then $\sigma(c_{k+1}) = \cdots = \sigma(c_n) = 0$.
- If $m \equiv 0 \pmod 2$ then $\sigma(c_1), \dots, \sigma(c_k) \in \{0, m/2\}$. We can reorder the ovals $c_1, \dots, c_k$ in such a way that
$$\sigma(c_1) = \cdots = \sigma(c_{k_0}) = 0, \quad \sigma(c_{k_0+1}) = \cdots = \sigma(c_k) = m/2,$$
where k_0 is the numbers of ovals of (P, τ) with the value of σ equal to 0.
- If $m \equiv 1 \pmod 2$ then $\sigma(c_1) = \cdots = \sigma(c_n) = 0$.

- We can choose bridges $d_1, \dots, d_{n-1}$ with the values $\sigma(d_i)$ as described in Lemma 3.5 since the assumptions of the Lemma are satisfied.
- If $\varepsilon = 1$ and $m \equiv 0 \pmod 2$, we can change the order of $c_1, \dots, c_{k_0}$ and $c_{k_0+1}, \dots, c_k$ to obtain the required values $\delta_1, \dots, \delta_{k-1}$.
- If $\varepsilon = 0$ and $m \equiv 0 \pmod 2$, there exists $\xi \in \{0, 1\}$ such that
$$\sigma(d_1) = \dots = \sigma(d_{n-1}) = \xi.$$
According to Theorem 4.9(4) in [NP16] the Arf invariant of σ is
$$\delta \equiv \sum_{i=1}^{n-1} (1 - \sigma(c_i))(1 - \sigma(d_i)) \pmod 2.$$
Using $\sigma(d_i) = \xi$ we obtain
$$\begin{aligned} \delta &\equiv \sum_{i=1}^{n-1} (1 - \sigma(c_i))(1 - \sigma(d_i)) \\ &\equiv (1 - \xi) \cdot \sum_{i=1}^{n-1} (1 - \sigma(c_i)) \\ &\equiv (1 - \xi) \cdot \left((n-1) - \sum_{i=1}^{n-1} \sigma(c_i) \right) \\ &\equiv (1 - \xi) \cdot \left((n-1) - k_1 \cdot \frac{m}{2} \right) \pmod 2. \end{aligned}$$
Recall that $k_1 \cdot m/2 \equiv 1 - g \pmod m$ by Proposition 3.2 and $n \equiv g - 1 \pmod 2$, hence
$$(n-1) - k_1 \cdot \frac{m}{2} \equiv (g-2) - (1-g) \equiv 2g - 3 \equiv 1 \pmod 2$$
and
$$\delta \equiv (1 - \xi) \cdot \left((n-1) - k_1 \cdot \frac{m}{2} \right) \equiv 1 - \xi \pmod 2.$$
Therefore
$$\sigma(d_1) = \dots = \sigma(d_{n-1}) = \xi = 1 - \delta.$$
- For $\tilde{g} \geqslant 2$, Lemma 5.1 in [NP09] implies that we can choose a standard generating set $(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n)$ of $\pi_1(P_1)$ so that
$$(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_{\tilde{g}}), \sigma(b_{\tilde{g}})) = (0, 1 - \tilde{\delta}, 1, \dots, 1),$$
where $\tilde{\delta}$ is the Arf invariant of $\sigma|_{P_1}$. Moreover, if m is odd then $\tilde{\delta} = 0$. If m is even and $\varepsilon = 0$ then there are closed curves around holes in P_1 such that the values of σ on these closed curves are even, namely $\sigma(c_{k+1}) = \dots = \sigma(c_n) = 0$, hence $\tilde{\delta} = 0$.
- If $\tilde{g} = 1$, Lemma 5.2 in [NP09] implies that we can choose a standard generating set $(a_1, b_1, c_1, \dots, c_n)$ of $\pi_1(P_1)$ so that
$$(\sigma(a_1), \sigma(b_1)) = (\tilde{\delta}, 0),$$
where $\tilde{\delta} = \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1)+1, \dots, \sigma(c_n)+1)$ is the Arf invariant of $\sigma|_{P_1}$. If m is odd then $\sigma(c_1) = \dots = \sigma(c_n) = 0$, hence $\tilde{\delta} = 1$. If $\varepsilon = 0$ then $\sigma(c_{k+1}) = \dots = \sigma(c_n) = 0$, hence $\tilde{\delta} = 1$.

□

Proposition 3.7. *For any Klein surface (P,τ) and any symmetric generating set $\mathcal{B}$ of $\pi_1(P)$ and any tuple t that satisfies the conditions of Proposition 3.2 there exists a real m-Arf function of the topological type t on (P,τ) for which $\mathcal{B}$ is canonical.*

Proof. Let $\mathcal{V} = (\alpha_i, \beta_i, \alpha'_i, \beta'_i, \gamma_i, \delta_i)$ satisfy the conditions of Definition 3.7.

- Case $\varepsilon = 0$, $m \equiv 0 \pmod 2$, $t = (g, \delta, k_0, k_1)$: We have $\gamma_1 = \cdots = \gamma_{k_0} = 0$, $\gamma_{k_0+1} = \cdots = \gamma_{k_0+k_1} = m/2$, hence $\sum_{i=1}^{k} \gamma_i = k_1 \cdot m/2$. The tuple t satisfies the conditions of Proposition 3.2, hence $k_1 \cdot m/2 \equiv 1 - g \pmod m$. Therefore $\sum_{i=1}^{k} \gamma_i \equiv 1 - g \pmod m$. Other conditions of Theorem 4.9(2) in [NP16] are clearly satisfied. Hence there exists a real m-Arf function σ on P with the values $\mathcal{V}$ on $\mathcal{B}$. If δ' is the Arf invariant of σ, then
$$\begin{aligned}\delta' &\equiv \sum_{i=1}^{n-1}(1-\gamma_i)(1-\delta_i) \equiv \sum_{i=1}^{n-1}(1-\gamma_i)(1-(1-\delta))\\ &\equiv \delta \cdot \sum_{i=1}^{n-1}(1-\gamma_i) \equiv \delta \cdot \left((n-1) - \sum_{i=1}^{n-1}\gamma_i\right)\\ &\equiv \delta \cdot \left((n-1) - k_1 \cdot \frac{m}{2}\right) \pmod 2.\end{aligned}$$
Recall that $k_1 \cdot m/2 \equiv 1 - g \pmod m$ and $n \equiv g - 1 \pmod 2$, hence
$$(n-1) - k_1 \cdot \frac{m}{2} \equiv (g-2) - (1-g) \equiv 2g - 3 \equiv 1 \pmod 2$$
and
$$\delta' \equiv \delta \cdot \left((n-1) - k_1 \cdot \frac{m}{2}\right) \equiv \delta \pmod 2.$$
Hence σ is a real m-Arf function on P of type t and $\mathcal{B}$ is canonical for σ.
- Case $\varepsilon = 1$, $m \equiv 0 \pmod 2$, $t = (g, \tilde{\delta}, \{(k_0^0, k_1^0), (k_0^1, k_1^1)\})$: The tuple t satisfies the conditions of Proposition 3.2, hence
$$1 - g \equiv k_1 \cdot \frac{m}{2} \pmod m$$
and therefore
$$1 - g \equiv 0 \pmod{\frac{m}{2}}.$$
Other conditions of Theorem 4.9(2) in [NP16] are clearly satisfied. Hence there exists a real m-Arf function σ on P with the values $\mathcal{V}$ on $\mathcal{B}$. Let $\tilde{\delta}'$ be the Arf invariant of $\sigma|_{P_1}$. The m-Arf function σ is real, hence according to Proposition 3.2, we have
  - If $g > k+1$ and $m \equiv 0 \pmod 4$ then $\tilde{\delta}' = 0$.
  - If $g > k+1$ and $k_0 \neq 0$ then $\tilde{\delta}' = 0$.
  - If $g = k+1$ and $m \equiv 0 \pmod 4$ then $\tilde{\delta}' = 1$.
  - If $g = k+1$ and $k_0 \neq 0$ then $\tilde{\delta}' = 1$.
  - If $g = k+1$, $m \equiv 2 \pmod 4$ and $k_0 = 0$ then $\tilde{\delta}' \in \{1, 2\}$.

  On the other hand $t = (g, \tilde{\delta}, \{(k_0^0, k_1^0), (k_0^1, k_1^1)\})$ satisfies the conditions of Proposition 3.2, hence
  - If $g > k+1$ and $m \equiv 0 \pmod 4$ then $\tilde{\delta} = 0$.
  - If $g > k+1$ and $k_0 \neq 0$ then $\tilde{\delta} = 0$.
  - If $g = k+1$ and $m \equiv 0 \pmod 4$ then $\tilde{\delta} = 1$.

- If $g=k+1$ and $k_0\neq 0$ then $\tilde{\delta}=1$.
- If $g=k+1$, $m\equiv 2 \pmod 4$ and $k_0=0$ then $\tilde{\delta}\in\{1,2\}$.

Hence if $m\equiv 0 \pmod 4$ or $k_0\neq 0$ we have $\tilde{\delta}'=\tilde{\delta}$. It remains to consider the case $m\equiv 2 \pmod 4$, $k_0=0$. In the case $g>k+1$, $m\equiv 2 \pmod 4$, $k_0=0$, we have $\tilde{g}\geqslant 2$ and the values of the m-Arf function $\sigma|_{P_1}$ on the boundary curves $\sigma(c_i)$ are all equal to $m/2$ and hence odd. Then, according to Theorem 4.4(c) in [NP16], the Arf invariant $\tilde{\delta}'$ is given by

$$\tilde{\delta}'\equiv\sum_{i=1}^{\tilde{g}}(1-\alpha_i)(1-\beta_i) \pmod 2.$$

We have $(\alpha_1,\beta_1,\dots,\alpha_{\tilde{g}},\beta_{\tilde{g}})=(0,1-\tilde{\delta},1,\dots,1)$, hence

$$\tilde{\delta}'\equiv\sum_{i=1}^{\tilde{g}}(1-\alpha_i)(1-\beta_i)\equiv 1\cdot\tilde{\delta}+0+\cdots+0\equiv\tilde{\delta} \pmod 2$$

and therefore $\tilde{\delta}'=\tilde{\delta}$. In the case $g=k+1$, $m\equiv 2 \pmod 4$, $k_0=0$, we have $\tilde{g}=1$ and the values of the m-Arf function $\sigma|_{P_1}$ on the boundary curves $\sigma(c_i)$ are all equal to $m/2$. Then, according to Theorem 4.4(d) in [NP16], the Arf invariant $\tilde{\delta}'\in\{1,2\}$ is given by

$$\tilde{\delta}'=\gcd\left(m,\alpha_1,\beta_1,\frac{m}{2}+1\right).$$

We have $(\alpha_1,\beta_1)=(\tilde{\delta},0)$, hence $\gcd(\alpha_1,\beta_1)=\tilde{\delta}\in\{1,2\}$. For $m\equiv 2 \pmod 4$ we have $\gcd\left(m,\frac{m}{2}+1\right)=2$. Therefore

$$\tilde{\delta}'=\gcd\left(m,\alpha_1,\beta_1,\frac{m}{2}+1\right)=\gcd(\tilde{\delta},2)=\tilde{\delta}.$$

Hence σ is a real m-Arf function on P of type t and $\mathcal{B}$ is canonical for σ.

- Case $m\equiv 1 \pmod 2$, $t=(g,k)$: The tuple t satisfies the conditions of Proposition 3.2, hence $g\equiv 1 \pmod m$. Other conditions of Theorem 4.10(2) in [NP16] are clearly satisfied. Hence there exists a real m-Arf function σ on P with the values $\mathcal{V}$ on $\mathcal{B}$. The topological type of σ is t and $\mathcal{B}$ is canonical for σ.

□

Proposition 3.8. *The conditions in Proposition 3.2 are necessary and sufficient for a tuple to be the topological type of a real m-Arf function.*

Proof. Proposition 3.2 shows that the conditions are necessary. Proposition 3.7 shows that the conditions are sufficient as we constructed an m-Arf function of type t for any tuple t that satisfies the conditions. □

Definition 3.8. Two m-Arf functions σ_1 and σ_2 on a Klein surface (P,τ) are *topologically equivalent* if there exists a homeomorphism $\varphi: P\to P$ such that $\varphi\circ\tau=\tau\circ\varphi$ and $\sigma_1=\sigma_2\circ\varphi_*$ for the induced automorphism φ_* of $\pi_1(P)$.

Proposition 3.9. *Let (P,τ) be a Klein surface of positive geometric genus. Two m-Arf functions on (P,τ) are topologically equivalent if and only if they have the same topological type.*

Proof. Let (g,k,ε) be the topological type of the Klein surface (P,τ). Proposition 3.6 shows that for any real m-Arf function σ of the topological type t we can choose a symmetric generating set $\mathcal{B}$ (the canonical generating set for σ) with the values of σ on $\mathcal{B}$ determined completely by t. Hence any two real m-Arf functions of the topological type t are topologically equivalent. $\square$

4. Moduli Spaces

In this section we will describe the space of real m-spin bundles as a branched covering of the space of underlying Klein surfaces (Theorem 4.3) and determine the branching indices (Theorem 4.4).

We will first recall the descriptions of the space of Klein surfaces and of the corresponding Teichmüller space, see [Nat75, Nat78]: We consider hyperbolic Klein surfaces, i.e. we assume that the genus is $g \geqslant 2$. Let $\mathcal{M}_{g,k,\varepsilon}$ be the moduli space of Klein surfaces of the topological type (g,k,ε). Let $\Gamma_{g,n}$ be the group generated by the elements

$$v = \{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n\}$$

with a single defining relation

$$\prod_{i=1}^{g} [a_i, b_i] \prod_{i=1}^{n} c_i = 1.$$

Let $\mathrm{Aut}_+(\mathbb{H})$ be the group of all orientation-preserving isometries of $\mathbb{H}$. The *Fricke space* $\tilde{T}_{g,n}$ is the set of all monomorphisms $\psi : \Gamma_{g,n} \to \mathrm{Aut}_+(\mathbb{H})$ such that

$$\{\psi(a_1), \psi(b_1), \dots, \psi(a_g), \psi(b_g), \psi(c_1), \dots, \psi(c_n)\}$$

is a generating set of a Fuchsian group of signature (g,n). The Fricke space $\tilde{T}_{g,n}$ is homeomorphic to $\mathbb{R}^{6g-3+3n}$. The group $\mathrm{Aut}_+(\mathbb{H})$ acts on $\tilde{T}_{g,n}$ by conjugation. The *Teichmüller space* is $T_{g,n} = \tilde{T}_{g,n}/\mathrm{Aut}_+(\mathbb{H})$.

Theorem 4.1. *Let (g,k,ε) be a topological type of a Klein surface. In the case $\varepsilon = 1$ let $n = k$. In the case $\varepsilon = 0$ we choose $n \in \{k+1, \dots, g+1\}$ such that $n \equiv g+1 \pmod 2$. Let $\tilde{g} = (g+1-n)/2$. The moduli space $\mathcal{M}_{g,k,\varepsilon}$ of Klein surfaces of the topological type (g,k,ε) is the quotient of the Teichmüller space $T_{\tilde{g},n}$ by a discrete group of autohomeomorphisms $\mathrm{Mod}_{g,k,\varepsilon}$. The space $T_{\tilde{g},n}$ is homeomorphic to $\mathbb{R}^{3g-3}$.*

Theorem 4.2. *The moduli space of Klein surfaces of genus g decomposes into connected components $\mathcal{M}_{g,k,\varepsilon}$. Each connected component is homeomorphic to the quotient of $\mathbb{R}^{3g-3}$ by a discrete group action.*

We will now state the main result of this paper, listing all connected components of the space of real m-spin bundles and describing their topology.

Theorem 4.3. *Let (g,k,ε) be a topological type of a Klein surface. Assume that the geometric genus of such Klein surfaces is positive, i.e. $k \leqslant g-2$ if $\varepsilon = 0$ and $k \leqslant g-1$ if $\varepsilon = 1$. Let t be a tuple that satisfies the conditions of Proposition 3.2. The space $S(t)$ of all m-spin bundles of type t on a Klein surface of type (g,k,ε) is connected and diffeomorphic to*

$$\mathbb{R}^{3g-3}/\mathrm{Mod}_t,$$

where Mod_t is a discrete group of diffeomorphisms.

Proof. In the case $\varepsilon = 1$ let $n = k$. In the case $\varepsilon = 0$ we choose $n \in \{k+1, \dots, g-1\}$ such that $n \equiv g-1 \pmod 2$. Let $\tilde{g} = (g+1-n)/2$. By definition, to any $\psi \in \tilde{T}_{\tilde{g},n}$ corresponds a generating set

$$V = \{\psi(a_1), \psi(b_1), \dots, \psi(a_{\tilde{g}}), \psi(b_{\tilde{g}}), \psi(c_1), \dots, \psi(c_n)\}$$

of a Fuchsian group of signature $(\tilde{g}, n)$. The generating set V together with

$$\{\overline{\psi(c_1)}, \dots, \overline{\psi(c_k)}, \widetilde{\psi(c_{k+1})}, \dots, \widetilde{\psi(c_n)}\}$$

generates a real Fuchsian group Γ_ψ. On the Klein surface $(P, \tau) = [\Gamma_\psi]$, we consider the corresponding symmetric generating set

$$\mathcal{B}_\psi = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, c_1, \dots, c_{n-1}, d_1, \dots, d_{n-1}).$$

Proposition 3.7 implies that there exists a real m-Arf function $\sigma = \sigma_\psi$ of type t for which $\mathcal{B}_\psi$ is canonical. According to Theorem 3.11 in [NP16], an m-spin bundle $\Omega(\psi) \in S(t)$ is associated with this m-Arf function. The correspondence $\psi \mapsto \Omega(\psi)$ induces a map $\Omega : T_{\tilde{g},n} \to S(t)$. Let us prove that $\Omega(T_{\tilde{g},n}) = S(t)$. Indeed, by Theorem 4.1, the map

$$\Psi = \Phi \circ \Omega : T_{\tilde{g},n} \to S(t) \to \mathcal{M}_{g,k,\varepsilon},$$

where Φ is the natural projection, satisfies the condition

$$\Psi(T_{\tilde{g},n}) = \mathcal{M}_{g,k,\varepsilon}.$$

The fibre of the map Ψ is represented by the group $\mathrm{Mod}_{g,k,\varepsilon}$ of all self-homeomorphisms of the Klein surface (P, τ). By Proposition 3.9, this group acts transitively on the set of all real m-Arf functions of type t and hence, by Theorem 3.11 in [NP16], transitively on the fibres $\Phi^{-1}((P, \tau))$. Thus

$$\Omega(T_{\tilde{g},n}) = S(t) = T_{\tilde{g},n}/\,\mathrm{Mod}_t, \quad \text{where } \mathrm{Mod}_t \subset \mathrm{Mod}_{g,k,\varepsilon}$$

According to Theorem 4.1, the space $T_{\tilde{g},n}$ is diffeomorphic to $\mathbb{R}^{3g-3}$. □

In the following theorem we determine the branching indices for the branched covering of the space of Klein surfaces by the space of real m-spin bundles.

Theorem 4.4. *Let (g, k, ε) be a topological type of a Klein surface. Assume that the geometric genus of such Klein surfaces is positive, i.e. $k \leqslant g-2$ if $\varepsilon = 0$ and $k \leqslant g-1$ if $\varepsilon = 1$. Let t be a tuple that satisfies the conditions of Proposition 3.2. The space $S(t)$ of all real m-spin bundles of type t on a Klein surface of type (g, k, ε) is an $N(t)$-fold covering of $\mathcal{M}_{g,k,\varepsilon}$, where $N(t)$ is the number of real m-Arf functions on (P, τ) of the topological type t. The number $N(t)$ is equal to*

1) Case $\varepsilon = 0$, $m \equiv 0 \pmod 2$, $t = (g, \delta, k_0, k_1)$:

$$N(t) = \binom{k}{k_1} \cdot \frac{m^g}{2}.$$

2) Case $\varepsilon = 1$, $m \equiv 0 \pmod 2$, $t = (g, \tilde{\delta}, \{(k_0^0, k_1^0), (k_0^1, k_1^1)\})$: Let

$$M = \binom{k}{k_0} \cdot \binom{k_0}{k_0^0} \cdot \binom{k_1}{k_1^0}.$$

- *Case $g > k+1$, ($m \equiv 0 \pmod 4$ or $k_0 \neq 0$):*

$$N(t) = 2^{1-k} \cdot m^g \cdot M \quad \text{for } \tilde{\delta} = 0 \text{ and } N(t) = 0 \quad \text{for } \tilde{\delta} = 1.$$

- *Case $g > k+1$, $m \equiv 2 \pmod 4$, $k_0 = 0$:*
$$N(t) = \left(2^{-k} + 2^{-\frac{g+k+1}{2}}\right) \cdot m^g \cdot M \quad \textit{for } \tilde{\delta} = 0,$$
$$N(t) = \left(2^{-k} - 2^{-\frac{g+k+1}{2}}\right) \cdot m^g \cdot M \quad \textit{for } \tilde{\delta} = 1.$$
- *Case $g = k+1$, ($m \equiv 0 \pmod 4$ or $k_0 \neq 0$):*
$$N(t) = 2^{-(k-1)} \cdot m^{k+1} \cdot M \quad \textit{for } \tilde{\delta} = 1 \textit{ and } N(t) = 0 \quad \textit{for } \tilde{\delta} = 2.$$
- *Case $g = k+1$, $m \equiv 2 \pmod 4$, $k_0 = 0$:*
$$N(t) = 3 \cdot 2^{-(k+1)} \cdot m^{k+1} \cdot M \quad \textit{for } \tilde{\delta} = 1,$$
$$N(t) = 2^{-(k+1)} \cdot m^{k+1} \cdot M \quad \textit{for } \tilde{\delta} = 2.$$

3) Case $m \equiv 1 \pmod 2$, $t = (g, k)$:
$$N(t) = m^g.$$

Proof. According to Theorem 4.3, $S(t) \cong T_{\tilde{g},n}/\operatorname{Mod}_t$, where $\operatorname{Mod}_t \subset \operatorname{Mod}_{g,k,\varepsilon}$, hence $S(t)$ is a branched covering of $\mathcal{M}_{g,k,\varepsilon} = T_{\tilde{g},n}/\operatorname{Mod}_{g,k,\varepsilon}$ and the branching index is equal to the index of the subgroup Mod_t in $\operatorname{Mod}_{g,k,\varepsilon}$, i.e. is equal to the number $N(t)$ of real m-Arf functions on (P, τ) of the topological type t. Let
$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, c_1, d_1, \dots, c_{n-1}, d_{n-1})$$
be a symmetric generating set of $\pi_1(P)$. Let $\mathcal{V} = (\alpha_i, \beta_i, \alpha'_i, \beta'_i, \gamma_i, \delta_i)$ denote the set of values of an m-Arf function on $\mathcal{B}$.

1) Case $\varepsilon = 0$, $m \equiv 0 \pmod 2$, $t = (g, \delta, k_0, k_1)$: There are $\binom{k}{k_1}$ ways to choose the values γ_i. There are $m^{2\tilde{g}}$ ways to choose $\alpha_i = \alpha'_i$ and $\beta_i = \beta'_i$. According to Theorem 4.9(5) in [NP16], out of m^{n-1} ways to choose $\delta_1, \dots, \delta_{n-1}$ there are $m^{n-1}/2$ which give $\Sigma \equiv 0 \pmod 2$ and $m^{n-1}/2$ which give $\Sigma \equiv 1 \pmod 2$. Thus the number of real m-Arf functions of type (g, δ, k_0, k_1) is
$$\binom{k}{k_1} \cdot m^{2\tilde{g}} \cdot \frac{m^{n-1}}{2} = \binom{k}{k_1} \cdot \frac{m^{2\tilde{g}+n-1}}{2} = \binom{k}{k_1} \cdot \frac{m^g}{2}.$$
2) Case $\varepsilon = 1$, $m \equiv 0 \pmod 2$, $t = (g, \tilde{\delta}, \{(k_0^0, k_1^0), (k_0^1, k_1^1)\})$: There are $M = \binom{k}{k_0} \cdot \binom{k_0}{k_0^0} \cdot \binom{k_1}{k_1^0}$ ways to choose the values γ_i. Moreover, for a fixed parity of δ_i, there are $(m/2)^{k-1}$ ways to choose the values of δ_i. Hence the number of such real m-Arf functions on P is equal to
$$m^{2\tilde{g}} \cdot \left(\frac{m}{2}\right)^{k-1} \cdot M = \frac{m^{2\tilde{g}+k-1}}{2^{k-1}} \cdot M = m^g \cdot 2^{1-k} \cdot M.$$
   - In the case $g > k+1$, $m \equiv 2 \pmod 4$, $k_0 = 0$, the resulting invariant $\tilde{\delta}$ is given by
$$\tilde{\delta} \equiv \sum_{i=1}^{\tilde{g}} (1 - \alpha_i)(1 - \beta_i) \pmod 2.$$
   It can be shown by induction that out of $m^{2\tilde{g}}$ ways to choose the values α_i, β_i we get the Arf invariant $\tilde{\delta} = 0$ in $2^{\tilde{g}-1}(2^{\tilde{g}} + 1)(m/2)^{2\tilde{g}}$ cases and $\tilde{\delta} = 1$ in $2^{\tilde{g}-1}(2^{\tilde{g}} - 1)(m/2)^{2\tilde{g}}$ cases. Hence the number $N(t)$ with $\tilde{\delta}$ equal to 0 and 1 respectively is
$$2^{\tilde{g}-1}(2^{\tilde{g}} \pm 1) \left(\frac{m}{2}\right)^{2\tilde{g}} \left(\frac{m}{2}\right)^{k-1} \cdot M.$$

We simplify

$$\begin{aligned}
&2^{\tilde g-1}(2^{\tilde g}\pm 1)\left(\frac{m}{2}\right)^{2\tilde g}\left(\frac{m}{2}\right)^{k-1}=(2^{2\tilde g-1}\pm 2^{\tilde g-1})\left(\frac{m}{2}\right)^{2\tilde g+k-1}\\
&=\left(2^{g-k}\pm 2^{\frac{g-k-1}{2}}\right)\left(\frac{m}{2}\right)^{g}=\left(2^{g-k}\pm 2^{\frac{g-k-1}{2}}\right)2^{-g}\cdot m^g\\
&=\left(2^{-k}\pm 2^{\frac{-g-k-1}{2}}\right)m^g=\left(2^{-k}\pm 2^{-\frac{g+k+1}{2}}\right)m^g
\end{aligned}$$

to obtain $N(t)$ as stated.

- In the case $g>k+1$, ($m\equiv 0 \pmod 4$ or $k_0\neq 0$), the Arf invariant of all m-Arf functions we construct is $\tilde\delta=0$, hence $N(t)$ is as stated.
- In the case $g=k+1$, $m\equiv 2\pmod 4$, $k_0=0$, the Arf invariant $\tilde\delta$ of the resulting m-Arf function is given by

$$\tilde\delta=\gcd\left(m,\alpha_1,\beta_1,\frac{m}{2}+1\right).$$

Note that for $m\equiv 2\pmod 4$ we have $\gcd(m,m/2+1)=2$, hence $\tilde\delta=2$ if α_1 and β_1 are both even and $\tilde\delta=1$ otherwise. Out of m^2 ways to choose the values α_1, β_1 we get $\tilde\delta=1$ in $3m^2/4$ cases and $\tilde\delta=2$ in $m^2/4$ cases. Hence the number $N(t)$ with $\tilde\delta$ equal to 1 and 2 respectively is

$$\frac{2\pm 1}{4}\cdot m^2\left(\frac{m}{2}\right)^{k-1}\cdot M=(2\pm 1)\cdot\left(\frac{m}{2}\right)^{k+1}\cdot M.$$

- In the case $g=k+1$, ($m\equiv 0\pmod 4$ or $k_0\neq 0$), the Arf invariant of all m-Arf functions we construct is $\tilde\delta=1$, hence $N(t)$ is as stated.

3) Case $m\equiv 1\pmod 2$, $t=(g,k)$: The statement follows from Theorem 4.10(3) in [NP16].

□

Example: Consider the case $g=3$, $m=4$. Let P be a compact Riemann surface of genus 3. According to Weichold's classification for a Klein surface (P,τ) either $\varepsilon=1$, $k\in\{2,4\}$ or $\varepsilon=0$, $k\in\{0,1,2,3\}$. Possible topological types of 4-spin bundles on these Klein surfaces are described in Propositions 3.2 and 3.8. Condition $k_1\cdot m/2\equiv 1-g\pmod m$ becomes $2k_1\equiv -2\pmod 4$ and is equivalent to k_1 being odd.

For example there exist 4-spin bundles on separating Klein surfaces (P,τ) with $k=2$ and for these bundles $k_0^0+k_0^1=k_1^0+k_1^1=1$ and $\tilde\delta=1$, i.e. the bundle is trivial on one of the ovals and non-trivial on the other and the 4-spin bundle restricted to $P\backslash P^\tau$ is odd. There are two possible topological types of such bundles:

$$(g,\tilde\delta,\{(k_0^0,k_1^0),(k_0^1,k_1^1)\})=(3,1,\{(1,1),(0,0)\}),$$
$$(g,\tilde\delta,\{(k_0^0,k_1^0),(k_0^1,k_1^1)\})=(3,1,\{(1,0),(0,1)\}).$$

Proposition 3.9 implies that 4-spin bundles on separating Klein surfaces of genus $g=3$ with $k=2$ are topologically equivalent if and only if they have the same topological type. Theorem 4.4 implies that the number $N(t)$ of real 4-spin bundles of the topological type $t=(g,\tilde\delta,\{(k_0^0,k_1^0),(k_0^1,k_1^1)\})$ is $N(t)=64$ for

$$t=(3,1,\{(1,1),(0,0)\})\quad\text{and}\quad t=(3,1,\{(1,0),(0,1)\}).$$

There exist 4-spin bundles on non-separating Klein surfaces (P,τ) with $k=1$ and for these bundles $k_0=0$, $k_1=1$, i.e. the bundle is non-trivial on the only oval. There are two possible topological types of such bundles:

$$(g,\delta,k_0,k_1)=(3,0,0,1) \quad \text{and} \quad (g,\delta,k_0,k_1)=(3,1,0,1).$$

Proposition 3.9 implies that 4-spin bundles on non-separating Klein surfaces of genus $g=3$ with $k=1$ are topologically equivalent if and only if they have the same topological type. Theorem 4.4 implies that the number $N(t)$ of real 4-spin bundles of the topological type $t=(g,\delta,k_0,k_1)$ is $N(t)=32$ for $t=(3,0,0,1)$ and $t=(3,1,0,1)$.

Similarly separating Klein surfaces (P,τ) with $k=4$ admit 4-spin bundles with topological types $(g=3,\tilde{\delta},\{(k_0^0,k_1^0),(k_0^1,k_1^1)\})$ with $(k_0,k_1)=(k_0^0+k_0^1,k_1^0+k_1^1)$ equal to $(1,3)$ or $(3,1)$ and non-separating Klein surfaces (P,τ) with $k=2,3$ admit 4-spin bundles of the topological types $(g=3,\tilde{\delta},\{(k_0^0,k_1^0),(k_0^1,k_1^1)\})$ with $(k_0^0+k_0^1,k_1^0+k_1^1)$ equal to $(1,1)$ or $(0,3)$ or $(2,1)$, while non-separating Klein surfaces (P,τ) with $k=0$ do not admit 4-spin structures. Geometric genus of Klein surfaces with $\varepsilon=1$, $k=4$ and $\varepsilon=0$, $k=2,3$ is equal to zero and their topological equivalence is not considered in this paper.

5. Applications in Singularity Theory

I. Dolgachev in [Dol75, Dol77, Dol83] described how all hyperbolic Gorenstein quasi-homogeneous surface singularities can be constructed by contracting the zero section of an m-spin bundle on $\mathbb{H}/\Gamma$ for some Fuchsian group Γ. (If the group Γ has torsion, a more careful construction using a normal torsion-free subgroup of Γ of finite index is necessary.) Hence a Klein surface structure on $\mathbb{H}/\Gamma$ leads to an anti-holomorphic involution on the singularity, i.e. to a real form of the singularity.

The correspondence between the weights of a quasi-homogeneous singularity and the signature of the Fuchsian group was studied in detail by K. Möhring in [Mo1, Mo2]. In this paper we only consider the case where Γ is a surface group, i.e. a Fuchsian group such that $\mathbb{H}/\Gamma$ is a compact Riemann surface. For general Gorenstein quasi-homogeneous surface singularities we need to consider Fuchsian groups Γ with torsion and m-spin bundles on the corresponding Klein orbifolds, i.e. on orbifolds $\mathbb{H}/\Gamma$ with an anti-holomorphic involution. The first results in this direction were obtained by Riley [Ril] who considered the case where the marked points of the orbifold $\mathbb{H}/\Gamma$ do not lie on the set of real points P^τ.

Let Γ be a Fuchsian group such that $\mathbb{H}/\Gamma$ is a compact Riemann surface of genus g. Let W be a corresponding weight system for a quasi-homogeneous singularity as described in [Mo1, Mo2]. Möhring states (Example 2.7 in [Mo2]) that among all quasi-homogeneous hypersurface singularities with the weight system W there is always a Brieskorn-Pham singularity, i.e. a singularity of the form $x^a+y^b+z^c=0$. Moreover, a (non-regular) normal quasi-homogeneous hypersurface singularity corresponds to a surface group if and only if it has the same weight system as a Brieskorn-Pham singularity $x^a+y^b+z^c=0$ such that no prime divides only one of the exponents a,b,c (see section 7 in [Mil75]).

For example 4-spin bundles on surfaces of genus 3 whose real forms were discussed above correspond to Brieskorn-Pham singularities $x^{14}+y^7+z^2=0$ and $x^{12}+y^4+z^3=0$ (Example 2.3 in [Mo2]). It would be of interest to make the connection between the anti-holomorphic involutions on a Riemann surface and on the corresponding singularities more explicit but this is beyond the scope of this paper.

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National Research University Higher School of Economics (HSE), Myasnitskaya Ulitsa 20, Moscow 101000, Russia

Institute of Theoretical and Experimental Physics (ITEP), B. Cheremushkinskaya 25, Moscow 117218, Russia
E-mail address: `natanzon@mccme.ru`

Department of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL
E-mail address: `annap@liv.ac.uk`